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2006 J. Phys. A: Math. Gen. 39 12129

(http://iopscience.iop.org/0305-4470/39/39/S08)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 12129–12151 doi:10.1088/0305-4470/39/39/S08

A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations

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Received 30 January 2006, in final form 25 July 2006
Published 13 September 2006
Online at stacks.iop.org/JPhysA/39/12129

Abstract

We revise Garnier–Okamoto's coalescent diagram of isomonodromic deformations and give a possible coalescent diagram from the viewpoint of isomonodromic deformations. We have ten types of isomonodromic deformations and two of them give the same type of Painlevé equation. We can naturally put the 34th Painlevé equation in our diagram, which corresponds to the Flaschka–Newell form of the second Painlevé equation.

PACS number: 02.30.Hq
Mathematics Subject Classification: 34M55

1. Introduction

In this paper, we revise Garnier–Okamoto's coalescent diagram of isomonodromic deformations [8] and give a possible coalescent diagram from the viewpoint of isomonodromic deformations. In the original form, the Painlevé equations are classified into six types. But in our picture, there exist ten different singularity types of isomonodromic deformations. Since two of them give the same type of Painlevé equation, the Painlevé equations are classified into eight different types.

We will also show that the Painlevé equations are classified into five types as a nonlinear single equation. In particular, we show a unified equation of the fourth Painlevé equation and the 34th Painlevé equation. These five types are classified into 14 types by scaling transformations. We exclude four types of them since they are quadrature. The remaining ten types of equations correspond to the different singularity types of isomonodromic deformations. In our form, it is easy to understand the relation between the type of Painlevé equation and the singularity type of isomonodromic deformation.

It is known that different forms of isomonodromic deformations

$$\frac{\partial Y}{\partial x} = A(x,t)Y, \qquad \frac{\partial Y}{\partial t} = B(x,t)Y$$

exist for some types of Painlevé equations. One of the most famous example is the Flaschka–Newell form [1] and the Miwa–Jimbo form [5] for the second Painlevé equation P2(α)

$$y'' = 2y^3 + ty + \alpha. \tag{1}$$

The Flaschka–Newell form (FN) is

$$\begin{aligned} A^{\mathrm{FN}}(x,t) &= -4\begin{pmatrix} x^2 & yx \\ yx & -x^2 \end{pmatrix} + \begin{pmatrix} t+2y^2 & -2z \\ 2z & -t-2y^2 \end{pmatrix} - \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\frac{1}{x}, \\ B^{\mathrm{FN}}(x,t) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}x + \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}. \end{aligned} \tag{2}$$

The Miwa–Jimbo form (MJ) is

$$\begin{aligned} A^{\mathrm{MJ}}(x,t) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}x^2 + \begin{pmatrix} 0 & u \\ \frac{-2}{u}z & 0 \end{pmatrix}x + \begin{pmatrix} z+\frac{t}{2} & -uy \\ \frac{-2}{u}(\theta+yz) & -z-\frac{t}{2} \end{pmatrix}, \\ B^{\mathrm{MJ}}(x,t) &= \frac{x}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & u \\ -\frac{2}{u}z & 0 \end{pmatrix}. \end{aligned} \tag{3}$$

We take a slightly different form from the original Flaschka–Newell form. Our form is a 'real' form in a sense. $A^{\mathrm{FN}}(x,t)$ has an irregular singularity of Poincaré rank 3 at $x = \infty$ and a regular singularity at $x = 0$. $A^{\mathrm{MJ}}(x,t)$ has an irregular singularity of Poincaré rank 3 but has no other singularities. They are not connected by any rational transform of the independent variable.

In this paper, we show that both MJ and FN come from different degenerations from the sixth Painlevé equation. Moreover, we will show that it is natural to consider that FN is a deformation for the 34th Painlevé equation P34(α) in Gambier's classification [3]

$$y'' = \frac{y'^2}{2y} + 2y^2 - ty - \frac{\alpha}{2y},$$

which is equivalent to the second Painlevé equation. Instead of the original P34, we change the sign $t \to -t$. We call this equation P34$'$.

It is known by Garnier and Okamoto that all types of Painlevé equations are represented as isomonodromic deformations of a single linear equation with order 2 [8]. MJ is essentially equivalent to the Garnier–Okamoto form. From the viewpoint of Garnier–Okamoto form, we obtain a well-known coalescent diagram of the Painlevé equations:

$$\begin{array}{ccccccccc} & & & \nearrow & (2+2) & \searrow & & & \\ (1+1+1+1) & \longrightarrow & (1+1+2) & & & & (4) & \longrightarrow & (7/2) \\ & & & \searrow & (1+3) & \nearrow & & & \end{array}$$

Here (j) is a pole order of the connection $A(x,t)$. This diagram is easy to understand and explains coalescence of the Painlevé equations [12]. But it seems that FN is out of the coalescent diagram since the type of singularities of FN is $(1+4)$. Later we will show our coalescent diagram of the Painlevé equations from the sixth Painlevé equation, which contains FN as type $(1+5/2)$.

Before we show the coalescent diagram, we will review the third Painlevé equation P3

$$y'' = \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2+\beta}{t} + \gamma y^3 + \frac{\delta}{y}. \tag{4}$$

P3 is divided into four types:

(P3-A) $\gamma \neq 0, \delta \neq 0$,

(P3-B) $\gamma \neq 0, \delta = 0$ or $\gamma = 0, \delta \neq 0$,

(P3-C) $\gamma = 0, \delta = 0$,

(P3-D) $\alpha = 0, \gamma = 0$ or $\beta = 0, \delta = 0$.

Since case P3-D is quadrature, we exclude case P3-D. Cases P3-A, P3-B and P3-C are called types $D_6^{(1)}$, $D_7^{(1)}$, $D_8^{(1)}$, respectively. The meaning of type is the Dynkin diagram of the intersection form of boundary divisors of the Okamoto initial value spaces [13]. In [9], we show that the corresponding linear equations for $D_7^{(1)}$ and $D_8^{(1)}$ types have singularities of types $(1)(1/2)$ and $(1/2)^2$. These three different types of the third equation are noted by Painlevé [11].

In the same way, the fifth Painlevé equation

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{1}{t} y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1} \tag{5}$$

has three types:

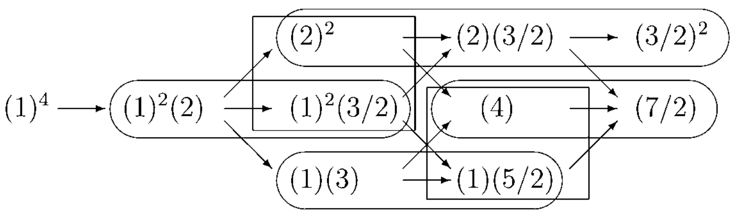
(P5-A) $\delta \neq 0$,

(P5-B) $\delta = 0, \gamma \neq 0$,

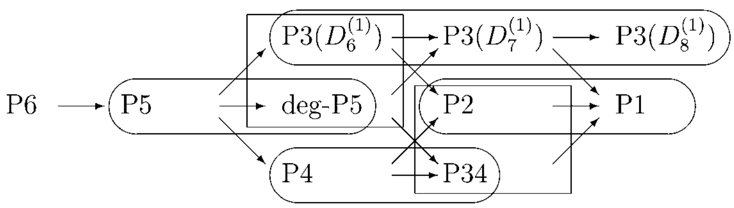
(P5-C) $\delta = 0, \gamma = 0$.

P5-A is a generic case. In case P5-B, the fifth Painlevé equation is equivalent to the third Painlevé equation of type $D_6^{(1)}$. In case P5-C, the fifth Painlevé equation is quadrature and we exclude case P5-C. We denote case P5-B as deg-P5.

In this paper, we study a possible coalescent diagram of singularity type:



The next diagram is the type of the Painlevé equation corresponding to the singularity diagram:



In both diagrams, we have two boxes and four ovals. We will show that the Painlevé equations in a box are equivalent (theorem 1). The Painlevé equations and their isomonodromic deformations in an oval can be unified in one equation (theorem 2).

We add four new types to old diagrams. All of them have a singularity whose order is a half-integer. Types $(1)^2(3/2)$, $(1)(5/2)$, $(2)(3/2)$ and $(3/2)^2$ correspond to deg-P5, P34, P3$\big(D_7^{(1)}\big)$ and P3$\big(D_8^{(1)}\big)$, respectively. The third Painlevé equation of $D_6^{(1)}$ type and the second Painlevé equation have two different types of isomonodromic deformations. Type $(1)^2(3/2)$

corresponds to the fifth Painlevé equation in the case $\delta = 0$. A transformation between FN and type $(1)(5/2)$ is also pointed out in [7]. We also add eight new arrows. We have two types of coalescences. The first is confluence of two singularities $(r_1)(r_2) \to (r_1 + r_2)$. The second is decrease in the Poincaré rank $(r) \to (r - 1/2)$ when $r = 2, 3, 4$. In the old diagram, the second type appeared only in the case P2 → P1.

Theorem 1. *The coalescent diagram which starts a linear differential equation with four regular singularities consists of ten types of singularities. We obtain eight different types of Painlevé equations from this diagram. The third Painlevé equation of $D_6^{(1)}$ type and the second Painlevé equation have two types of isomonodromic deformations.*

The first Painlevé equation P1

$$y'' = 6y^2 + t$$

can be considered as deg-P2. Painlevé showed a unified equation of P1 and P2 [10]:

$$y'' = \alpha(2y^3 + ty) + \beta(6y^2 + t). \tag{6}$$

In [10], Painlevé took $\beta = 1$. If $\alpha = 0$, (6) is nothing but P1. We will show that (6) is equivalent to P2 if $\alpha \neq 0$ in section 2.

deg-P5 is also a special case of P5, and P3$\left(D_7^{(1)}\right)$ and P3$\left(D_8^{(1)}\right)$ are also special cases of P3. In section 2, we show that equation P4_34$'(\alpha, \beta, \gamma)$

$$y'' = \frac{y'^2}{2y} - \frac{\alpha}{2y} + \beta y(2y + t) + \gamma y(y + t)(3y + t) \tag{7}$$

is a unified equation of P4 and P34$'$. If $\gamma = 0$, P4_34$'(\alpha, \beta, \gamma)$ is equivalent to P34$''$. If $\gamma \neq 0$, P4_34$'(\alpha, \beta, \gamma)$ is equivalent to P4. The authors cannot find the unified equation (7) in the literature.

Thus, we obtain the following observation.

Theorem 2. *In the coalescent diagram, equations in an oval can be represented as one unified equation. P5 and deg-P5 are unified as the standard fifth Painlevé equation. $P\left(D_6^{(1)}\right), P\left(D_7^{(1)}\right)$ and $P\left(D_8^{(1)}\right)$ are unified as the standard third Painlevé equation. P4 and P34 are unified as (7). P1 and P2 are unified as (6). In an oval, coalescence reduces the Poincaré rank of a singularity by 1/2. The corresponding linear equations are also unified in one unified equation.*

As a single nonlinear equation, the Painlevé equations are classified into five types. Each type has a scaling transformation $t \to c_1 t$, $y \to c_2 y$ except P6. We obtain eight types of Painlevé equations after we classify again each type by the scaling transformation,

In section 2, we review the Painlevé equations. We will show that the Painlevé equations in the same box are equivalent. In section 3, we show that FN comes from an isomonodromic deformation of type $(1)(5/2)$.

We will give two types of isomonodromic deformations of the Painlevé equations. The first form is called the canonical type and the second form is called SL-type. We study the canonical type in section 4. This form is easy to study when we consider ten types of Painlevé equations, and most of the natural Hamiltonians are polynomials. In section 4.2, we give a degeneration of the extended linear equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + p(x,t)\frac{\mathrm{d}u}{\mathrm{d}x} + q(x,t)u = 0, \qquad \frac{\partial u}{\partial t} = a(x,t)\frac{\partial u}{\partial x} + b(x,t)u$$

of the Painlevé equations.

We study SL-type in section 5. In this form, the extended linear equations of the Painlevé equations in the same oval are also unified in one linear equation. But the Hamiltonians are not polynomials in this form. Most of the equations and degenerations are already listed in [8], but we correct misprints in [8]. Okamoto did not write $b(x,t)$ in the extended linear equation explicitly and did not give degeneration of the extended equation. We will show all the extended linear equations and their degenerations.

The authors thank Professor Hiroyuki Kawamuko for fruitful discussions.

2. List of the Painlevé equations

In this section, we list up the Painlevé equations in an unusual way. This classification is essential for our coalescent diagram. We also give some equivalence between different types of Painlevé equations. We will give a proof of the second part of theorem 1, although this is well known.

We list *five* types of Painlevé equations:

$$\text{(P1\_2)}\qquad y'' = \alpha(2y^3 + ty) + \beta(6y^2 + t),$$

$$\text{(P4\_34}'\text{)}\qquad y'' = \frac{y'^2}{2y} - \frac{\alpha}{2y} + \beta y(2y+t) + \gamma y(y+t)(3y+t),$$

$$\text{(P3)}\qquad y'' = \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y},$$

$$\text{(P5)}\qquad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1},$$

$$\text{(P6)}\qquad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right].$$

Here $\alpha, \beta, \gamma, \delta$ are complex parameters. $P1\_2$, $P4\_34'$, P3 and P5 have a scaling transformation. We will classify five types into 14 types by scaling transformations.

2.1. Unified equation of P1 and P2

By the scaling transformation $y \to cy, t \to c^2 t$, P1_2(α, β) is changed to P1_2$(c^6\alpha, c^5\beta)$. P1_2(α, β) is divided into three types:

(P1-A) $\alpha \neq 0$,

(P1-B) $\alpha = 0, \beta \neq 0$,

(P1-C) $\alpha = 0, \beta = 0$.

Lemma 3. *Case P1-A is equivalent to P2 and case P1-B is equivalent to P1:*

(P1) $\quad y'' = 6y^2 + t$,

(P2) $\quad y'' = 2y^3 + ty + \alpha$.

Case P1-C is trivial.

Proof. In case P1-B, we can set $\beta = 1$ by a scaling transformation and P1_2(0, 1) is nothing but P1. In case P1-A, we set $\alpha = \varepsilon^6$ and change the variables

$$y \to y\varepsilon^{-1} - \beta\varepsilon^{-6}, \qquad t \to t\varepsilon^{-2} + 6\beta^2\varepsilon^{-12}.$$

Then we obtain P2

$$y'' = 2y^3 + ty + \frac{4\beta^3}{\varepsilon^{15}}.$$

Therefore, P1_2(ε^6, β) is equivalent to P2($4\beta^3\varepsilon^{-15}$). □

2.2. Unified equation of P34 and P4

By the scaling transformation $y \to cy, t \to ct$, P4_34$'(\alpha, \beta, \gamma)$ is changed to P4_34$'(\alpha, c^3\beta, c^4\gamma)$. P4_34$'(\alpha, \beta, \gamma)$ is divided into three types:

(P4-A) $\gamma \neq 0$,

(P4-B) $\beta \neq 0, \gamma = 0$,

(P4-C) $\beta = 0, \gamma = 0$.

Lemma 4. *Case P4-A is equivalent to P4 and case P4-B is equivalent to P34:*

$$\text{(P34}'\text{)} \quad y'' = \frac{y'^2}{2y} + 2y^2 + ty - \frac{\alpha}{2y},$$

$$\text{(P4)} \quad y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}.$$

P2 and P34$'$ are equivalent. Case P4-C is quadrature.

Proof. In case P4-C, P4_34$'(\alpha, 0, 0)$ has a solution

$$y = C_1t^2 + C_2t + \frac{C_2^2 - \alpha}{4C_1}.$$

In case P4-B, we can set $\beta = 1$ by a scaling transformation and P4_34$'(\alpha, 1, 0)$ is nothing but P34$'(\alpha)$. In case P4-A, we set $\beta = d^3, \gamma = 2\varepsilon^4$ and change the variables

$$y \to \frac{y}{2\varepsilon}, t \to \varepsilon^{-1}t - \frac{d^3}{4\varepsilon^4}, \alpha \to -\beta/2.$$

Then we obtain P4($d^6\varepsilon^{-6}/16, \beta$)

$$y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2t^2y - \frac{d^6y}{8\varepsilon^6} + \frac{\beta}{y}.$$

We will show the equivalence between P2 and P34$'$. The second Painlevé equation (1) is represented by a Hamiltonian form

$$\mathcal{H}_{\text{II}} : \begin{cases} q' = -q^2 + p - \dfrac{t}{2}, \\ p' = 2pq + a, \end{cases} \tag{8}$$

with the Hamiltonian

$$H_{\text{II}} = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - aq.$$

If we remove p from (8), we obtain P2($a - 1/2$). If we remove q from (8), we obtain P34(a^2). Therefore, P2 and P34 are equivalent. P34 and P34$'$ are equivalent by $t \to -t$.

More precisely, if y satisfies the second Painlevé equation P2(α), the function $p = y^2 + y' + t/2$ satisfies P34($(\alpha + 1/2)^2$). Conversely, if p satisfies P34(α), $q = \frac{1}{2p}(p' - \sqrt{\alpha})$ satisfies P2($\sqrt{\alpha} - 1/2$). □

Remark. If we choose $-\sqrt{\alpha}$ instead of $\sqrt{\alpha}$, we obtain $\mathrm{P2}(-\sqrt{\alpha}-1/2)$ which is equivalent to $\mathrm{P2}(\sqrt{\alpha}-1/2)$ by a Bäcklund transformation. The equivalence of P2 and P34 is known by [3].

We will use P34′ instead of P34. If we change the sign of t, we obtain a canonical transformation $(p, q, H, t) \to (q, p, H, -t)$:

$$\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t = -\left(\mathrm{d}q \wedge \mathrm{d}p - \mathrm{d}H \wedge \mathrm{d}(-t)\right).$$

In the following, we may use $\sigma = \pm 1$ to express both P34 and P34′:

$$y'' = \frac{y'^2}{2y} + 2y^2 + \sigma t y - \frac{\alpha}{2y}.$$

Similarly, we express both P4_34 and P4_34′:

$$y'' = \frac{y'^2}{2y} - \frac{\alpha}{2y} + \beta y(2y + \sigma t) + \gamma y(y + \sigma t)(3y + \sigma t).$$

2.3. P3

By the scaling transformation $y \to c_1 y, t \to c_2 t$, $\mathrm{P3}(\alpha, \beta, \gamma, \delta)$ is changed to $\mathrm{P3}\left(c_1 c_2 \alpha, c_2/c_1 \beta, c_1^2 c_2^2 \gamma, c_2^2/c_1^2 \delta\right)$. $\mathrm{P3}(\alpha, \beta, \gamma, \delta)$ is divided into four types:

(P3-A) $\gamma \neq 0, \delta \neq 0$,

(P3-B) $\gamma \neq 0, \delta = 0$ or $\gamma = 0, \delta \neq 0$,

(P3-C) $\gamma = 0, \delta = 0$,

(P3-D) $\alpha = 0, \gamma = 0$ or $\beta = 0, \delta = 0$.

P3-A is $\mathrm{P3}\left(D_6^{(1)}\right)$, P3-B is $\mathrm{P3}\left(D_7^{(1)}\right)$, P3-C is $\mathrm{P3}\left(D_8^{(1)}\right)$ and P3-D is quadrature. As usual, we fix $\gamma = 4, \delta = -4$ for $\mathrm{P3}\left(D_6^{(1)}\right)$, $\alpha = 2, \gamma = 0, \delta = -4$ for $\mathrm{P3}\left(D_7^{(1)}\right)$ and $\alpha = 4, \beta = -4, \gamma = 0, \delta = 0$ for $\mathrm{P3}\left(D_8^{(1)}\right)$. See [9].

We will use another form of the third Painlevé equation $\mathrm{P3}'(\alpha, \beta, \gamma, \delta)$

$$q'' = \frac{1}{q} q'^2 - \frac{q'}{x} + \frac{\alpha q^2 + \gamma q^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4q},$$

since P3′ is more sympathetic to isomonodromic deformations than P3. We can change P3 to P3′ by $x = t^2, ty = q$.

2.4. P5

By the scaling transformation $t \to ct$, $\mathrm{P5}(\alpha, \beta, \gamma, \delta)$ is changed to $\mathrm{P5}(\alpha, \beta, c\gamma, c^2\delta)$. $\mathrm{P5}(\alpha, \beta, \gamma, \delta)$ is divided into three types:

(P5-A) $\delta \neq 0$,

(P5-B) $\delta = 0, \gamma \neq 0$,

(P5-C) $\delta = 0, \gamma = 0$.

Case P5-A is a generic P5 and we call P5-B as deg-P5. As usual, we fix $\delta = -1/2$ for P5-A and $\gamma = -2, \delta = 0$ for P5-B.

Lemma 5. *P5-B is equivalent to* $\mathrm{P3}'\left(D_6^{(1)}\right)$ *and P5-C is quadrature.*

Proof. $\mathrm{P3}'\big(D_6^{(1)}\big)$ is represented by a Hamiltonian form

$$\mathcal{H}'_{D_6}:\begin{cases} tq' = 2pq^2 - q^2 + (\alpha_1+\beta_1)q + t, \\ tp' = -2p^2q + 2pq - (\alpha_1+\beta_1)p + \alpha_1, \end{cases} \tag{9}$$

with the Hamiltonian

$$tH'_{D_6} = q^2p^2 - (q^2 - (\alpha_1+\beta_1)q - t)p - \alpha_1 q.$$

If we eliminate p from (9), q satisfies $\mathrm{P3}'(4(\alpha_1-\beta_1), -4(\alpha_1+\beta_1-1), 4, -4)$. If we eliminate q from (9) and set $y = 1 - 1/p$, y satisfies deg-P5$\big(\alpha_1^2/2, -\beta_1^2/2, -2, 0\big)$. We can write y directly by q:

$$y = \frac{tq' - q^2 - (\alpha_1+\beta_1)q - t}{tq' + q^2 - (\alpha_1+\beta_1)q - t}.$$

Therefore, deg-P5 is equivalent to $\mathrm{P3}\big(D_6^{(1)}\big)$. This is known by [4]. □

2.5. Summary

If we classify the five types of Painlevé equations by scaling transformations, we obtain 14 types of equations. Four of them are quadrature. Thus, we have *ten* types of Painlevé equations:

(P1-A), (P1-B), (P4-A), (P4-B), (P3-A), (P3-B), (P3-C), (P5-A), (P5-B), (P6).

P1-A and P4-B are equivalent and P3-A and P5-B are equivalent.

3. The Flaschka–Newell form and P34

In this section, we will prove that FN comes from isomonodromic deformations of type $(1)(5/2)$ and show that it natural to consider the Flaschka–Newell form as an isomonodromic deformation of P34 not of P2. This proves the rest part of theorem 1. The relation between the Flaschka–Newell form and P34 was noted by Kapaev and Hubert [6, 7].

First, we will review the Poincaré rank of irregular singularities. We consider a linear equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} + p_2(x)u = 0. \tag{10}$$

Assume that

$$p_1(x) = c_0x^k + c_1x^{k-1} + \cdots, \qquad p_2(x) = d_0x^l + d_1x^{l-1} + \cdots$$

around $x = \infty$ and c_0, d_0 are not zero. If

$$r = \max(k+1, (l+2)/2)$$

is positive, $x = \infty$ is an irregular singularity of (10). We call r as the Poincaré rank of (10) at $x = \infty$. The Poincaré rank r may be a half-integer. If $x = \infty$ is an irregular singularity with the Poincaré rank r, (10) has solutions with an asymptotic

$$u_j \sim \exp(\kappa_j x^r).$$

Proposition 6. *The Flaschka–Newell form of P2 is a double cover of a linear equation of singularity type* $(1)(5/2)$. *If we write the equation of type* $(1)(5/2)$ *as a single equation, the apparent singularity satisfies P34.*

Proof. We consider the following deformation equation:

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}w} &= \left[\begin{pmatrix}0 & 2w\\ 0 & 0\end{pmatrix} + \begin{pmatrix}-2y & -y^2-z-t/2\\ 2 & 2y\end{pmatrix} + \begin{pmatrix}-\alpha+1/2 & 0\\ -2y^2+2z-t & \alpha-1/2\end{pmatrix}\frac{1}{2w}\right]Z,\\ \frac{\partial Z}{\partial t} &= \begin{pmatrix}y & -w\\ -1 & -y\end{pmatrix}Z.\end{aligned} \tag{11}$$

By the compatibility condition, we obtain P2(α)

$$y' = z,\ z' = 2y^3 + ty + \alpha.$$

If we change $w = x^2$ and $Z = RY$ where

$$R = \begin{pmatrix}\sqrt{x} & \sqrt{x}\\ -1/\sqrt{x} & 1/\sqrt{x}\end{pmatrix},$$

we obtain the FN form (2). Since the exponents of (11) at $w = \infty$ coincide, the Poincaré rank at $w = \infty$ in (11) is (3/2). We will rewrite (11) as a single equation of the second order.

We change the variables

$$w \to \frac{w}{2},\ z \to p^2 + q - \frac{t}{2},\ y \to -p.$$

Then (11) is changed to

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}w} &= \left[\left\{\begin{pmatrix}p & x/2-q/2\\ 1 & -p\end{pmatrix}\right\} + \begin{pmatrix}-\alpha/2+1/4 & 0\\ q-2p^2-t & \alpha/2-1/4\end{pmatrix}\frac{1}{w}\right]Z,\\ \frac{\partial Z}{\partial t} &= \begin{pmatrix}-p & -x/2\\ -1 & p\end{pmatrix}Z.\end{aligned} \tag{12}$$

The compatibility condition is

$$\begin{cases} q' = -2pq + \left(\alpha + \dfrac{1}{2}\right),\\ p' = p^2 - q + \dfrac{t}{2}.\end{cases} \tag{13}$$

We set

$$Z = \begin{pmatrix}u_1\\ u_2\end{pmatrix},\ u_1 = w^{1/4-\alpha/2}u.$$

Eliminating u_2 from (12), we get a single equation for $u = u_1$:

$$\begin{aligned}&\frac{\mathrm{d}^2u}{\mathrm{d}w^2} + p_1(w,t)\frac{\mathrm{d}u}{\mathrm{d}w} + p_2(w,t)u = 0,\\ &\frac{\partial u}{\partial t} = a(w,t)\frac{\partial u}{\partial w} + b(w,t)u,\end{aligned} \tag{14}$$

where

$$p_1(w,t) = -\frac{1}{w-q} + \frac{1/2-\alpha}{w},\ p_2(w,t) = -\frac{w}{2} + \frac{t}{2} + \frac{\mathcal{H}_{34}}{w} + \frac{pq}{w(w-q)},$$

$$a(w,t) = -\frac{w}{w-q},\ b(w,t) = \frac{pq}{w-q},$$

$$\mathcal{H}_{34} = -qp^2 + \left(\alpha + \frac{1}{2}\right)p + \frac{q^2}{2} - \frac{1}{2}tq.$$

The isomonodromic deformation is equivalent to the Hamiltonian system (13) with the Hamiltonian $\mathcal{H}_{34}$. If we eliminate p from (13), we obtain P34($(\alpha + 1/2)^2$) for q.

The first equation of (14) has an regular singularity $w = 0$ and an irregular singularity of the Poincaré rank $3/2$ at $w = \infty$. It also has an apparent singularity $w = q$. When we write the Painlevé equations as isomonodromic deformations of linear equations of the second order, they have an apparent singularity, and the apparent singularity is the Painlevé function. Moreover,

$$p = \mathrm{Res}_{w=q}\, p_2(w, t)$$

is a canonical coordinate [8]. In the Flaschka–Newell case, the apparent singularity q satisfies P34 but not P2. □

4. Isomonodromic deformations of canonical type

This section is the revision of section 4.3 in [8]. In this section, we list up isomonodromic deformations of the canonical type L_J:

$$\frac{\partial^2 u}{\partial x^2} + p(x,t)\frac{\partial u}{\partial x} + q(x,t)u = 0, \qquad \frac{\partial u}{\partial t} = a(x,t)\frac{\partial u}{\partial x} + b(x,t)u. \tag{15}$$

The extended linear equation L_J is called the canonical type if it is obtained from the canonical type equation L_{VI} by the process of step-by-step confluence, and the Fuchsian equation L_{VI} is called the canonical type if either of the local exponents at any singular point is zero. The compatibility condition of (15) is

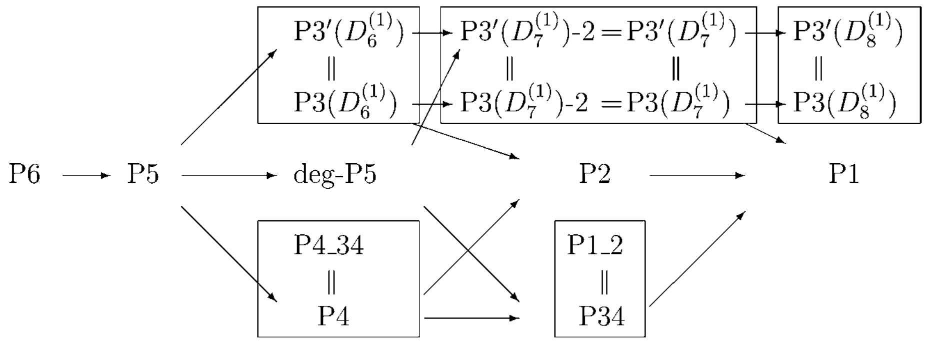
$$p_t(x,t) - (p(x,t)a(x,t))_x + a_{xx}(x,t) + 2b_x(x,t) = 0,$$
$$q_t(x,t) - 2q(x,t)a_x(x,t) - q_x(x,t)a(x,t) + p(x,t)b_x(x,t) + b_{xx}(x,t) = 0.$$

The second equation is an essential deformation equation and is a Hamiltonian system with the Hamiltonian $\mathcal{H}_J$. $b(x,t)$ is determined from the first equation by integration and if we change $b(x,t) \to b(x,t)+s(t)$, the compatibility condition is also satisfied. We can eliminate $s(t)$ by the transformation $u \to u \exp \int s(t)\,\mathrm{d}t$. In the following, we may change $b(x,t)$ up to an additive term $s(t)$.

4.1. List of canonical type

We have *ten* types of isomonodromic deformations: P1, P2, P34, P3$'\big(D_6^{(1)}\big)$, P3$'\big(D_7^{(1)}\big)$, P3$'\big(D_8^{(1)}\big)$, P4, P5, deg-P5 and P6. We will show isomonodromic deformation not only for P3$'$ but also for the original P3. We need two types of P3$'\big(D_7^{(1)}\big)$ for degeneration from deg-P5. One is the case $\gamma = 0$ and the other is the case $\delta = 0$. We also show the isomonodromic deformations for P1_2 and P4_34. But these unified equations are not necessary for degenerations.

We will list up 17 types, but they are classified into ten types up to algebraic transformations. We will show the degeneration diagram



Here double lines mean algebraic transformations and equations in a box are equivalent to each other.

Painlevé 1_2:

$$
\begin{aligned}
p(x,t) &= -2\eta x^2 - \eta t - \frac{1}{x-y},\\
q(x,t) &= -4\beta x^3 - (\eta + 2\beta t)x - 2\mathcal{H}_{1\_2} + \frac{z}{x-y},\\
a(x,t) &= \frac{1}{2(x-y)},\\
b(x,t) &= \frac{\beta\eta^{-1} - \eta y}{2} - \frac{z}{2(x-y)},\\
\mathcal{H}_{1\_2} &= \frac{1}{2}z^2 - \left(\eta y^2 + \frac{1}{2}\eta t\right)z - 2\beta y^3 - t\beta y - \frac{1}{2}\eta y,\\
\alpha &= \eta^2.
\end{aligned}
$$

Painlevé I:

$$
\begin{aligned}
p(x,t) &= -\frac{1}{x-y},\\
q(x,t) &= -4x^3 - 2tx - 2\mathcal{H}_I + \frac{z}{x-y},\\
a(x,t) &= \frac{1}{2(x-y)},\\
b(x,t) &= -\frac{z}{2(x-y)},\\
\mathcal{H}_I &= \frac{1}{2}z^2 - 2y^3 - ty.
\end{aligned}
$$

Painlevé II:

$$
\begin{aligned}
p(x,t) &= -2x^2 - t - \frac{1}{x-y},\\
q(x,t) &= -(2\alpha + 1)x - 2\mathcal{H}_{\rm II} + \frac{z}{x-y},\\
a(x,t) &= \frac{1}{2(x-y)},
\end{aligned}
$$

$$b(x,t) = -\frac{y}{2} - \frac{z}{2(x-y)},$$
$$\mathcal{H}_{\mathrm{II}} = \frac{1}{2}z^2 - \left(y^2 + \frac{1}{2}t\right)z - \left(\alpha + \frac{1}{2}\right)y.$$

Painlevé 4_34: if $\sigma = +1$, this gives P4_34′. If $\sigma = -1$, this gives P4_34.

$$p(x,t) = \theta - \sigma\eta t + \frac{1-\kappa_0}{x} - \eta x - \frac{1}{x-y},$$
$$q(x,t) = \frac{\theta^2}{4} + \frac{(\kappa_0 - 1)\eta}{2} - \frac{\sigma\mathcal{H}_{4\_34}}{x} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{\sigma x}{x-y},$$
$$b(x,t) = -\frac{\sigma yz}{x-y},$$
$$\mathcal{H}_{4\_34} = \sigma yz^2 - \sigma\left(\eta y^2 - \theta y + \kappa_0\right)z + \sigma\left(\frac{\theta^2}{4} + \frac{(\kappa_0-1)\eta}{2}\right)y - \eta tyz,$$
$$\alpha = \kappa_0^2, \beta = -\eta\theta, \gamma = \frac{1}{2}\eta^2.$$

Painlevé 34: if $\sigma = +1$, this gives P34′. If $\sigma = -1$, this gives P34.

$$p(x,t) = -\frac{1}{x-y} + \frac{1-\kappa_0}{x},$$
$$q(x,t) = -\frac{x}{2} - \frac{\sigma t}{2} - \frac{\sigma\mathcal{H}_{34}}{x} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{\sigma x}{x-y},$$
$$b(x,t) = -\frac{\sigma yz}{x-y},$$
$$\mathcal{H}_{34} = \sigma\left(yz^2 - \kappa_0 z - \frac{y^2}{2}\right) - \frac{1}{2}ty,$$
$$\alpha = \kappa_0^2.$$

Painlevé IV:

$$p(x,t) = \frac{1-\kappa_0}{x} - \frac{x+2t}{2} - \frac{1}{x-y},$$
$$q(x,t) = \frac{1}{2}\theta_\infty - \frac{\mathcal{H}_{\mathrm{IV}}}{2x} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{2x}{x-y},$$
$$b(x,t) = -\frac{1}{2}(y+2t) - \frac{2yz}{x-y}.$$
$$\mathcal{H}_{\mathrm{IV}} = 2yz^2 - (y^2 + 2ty + 2\kappa_0)z + \theta_\infty y,$$
$$\alpha = -\kappa_0 + 2\theta_\infty + 1, \beta = -2\kappa_0^2.$$

Painlevé III$\big(D_6^{(1)}\big)$:

$$p(x,t) = \frac{\eta_0 t}{x^2} + \frac{1-\theta_0}{x} - \eta_\infty t - \frac{1}{x-y},$$
$$q(x,t) = \frac{\eta_\infty(\theta_0+\theta_\infty)t}{2x} - \frac{t\mathcal{H}_{\rm III} + yz}{2x^2} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{2yx}{t(x-y)} + \frac{x}{t},$$
$$b(x,t) = -\eta_\infty y - \frac{2y^2 z}{t(x-y)},$$
$$t\mathcal{H}_{\rm III} = 2y^2z^2 - \{2\eta_\infty t y^2 + (2\theta_0+1)y - 2\eta_0 t\}z + \eta_\infty(\theta_0+\theta_\infty)ty,$$
$$\alpha = -4\eta_\infty\theta_\infty,\ \beta = 4\eta_0(1+\theta_0),\ \gamma = 4\eta_\infty^2,\ \delta = -4\eta_0^2.$$

Painlevé III$\big(D_7^{(1)}\big)$: the case $\gamma = 0$.

$$p(x,t) = \frac{\eta_0 t}{x^2} + \frac{1-\theta_0}{x} - \frac{1}{x-y},$$
$$q(x,t) = \frac{\theta_\infty t}{2x} - \frac{t\mathcal{H}_{D_7} + yz}{2x^2} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{2yx}{t(x-y)} + \frac{x}{t},$$
$$b(x,t) = -\frac{2y^2 z}{t(x-y)},$$
$$t\mathcal{H}_{D_7} = 2y^2z^2 - \{(2\theta_0+1)y - 2\eta_0 t\}\,z + \theta_\infty\,ty,$$
$$\alpha = -4\theta_\infty,\ \beta = 4(\theta_0+1)\eta_0,\ \gamma = 0,\ \delta = -4\eta_0^2.$$

Painlevé III$\big(D_7^{(1)}\big)$-2: the case $\delta = 0$.

$$p(x,t) = -\eta_\infty t + \frac{1}{x} - \frac{1}{x-y},$$
$$q(x,t) = \frac{\theta_0 t}{2x^3} - \frac{t\mathcal{H}_{D_7\text{-}2} + yz}{2x^2} + \frac{\theta_\infty\eta_\infty t}{2x} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{2yx}{t(x-y)} + \frac{x}{t},$$
$$b(x,t) = -\frac{2y^2 z}{t(x-y)},$$
$$t\mathcal{H}_{D_7\text{-}2} = 2y^2z^2 - \{2\eta_\infty t y^2 + y\}z + \theta_\infty\eta_\infty\,ty + \frac{\theta_0 t}{y},$$
$$\alpha = -4\theta_\infty\eta_\infty,\ \beta = 4\theta_0,\ \gamma = 4\eta_\infty^2,\ \delta = 0.$$

Painlevé III$\big(D_8^{(1)}\big)$:

$$p(x,t) = \frac{2}{x} - \frac{1}{x-y},$$
$$q(x,t) = \frac{t}{2x^3} - \frac{t\mathcal{H}_{D_8} + yz}{2x^2} + \frac{t}{2x} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{2yx}{t(x-y)} + \frac{x}{t},$$

$$b(x,t) = -\frac{2y^2z}{t(x-y)},$$
$$t\mathcal{H}_{D_8} = 2y^2z^2 + yz + ty + \frac{t}{y},$$
$$\alpha = -4, \beta = 4, \gamma = 0, \delta = 0.$$

Painlevé III$'\big(D_6^{(1)}\big)$:

$$p(x,t) = \frac{\eta_0 t}{x^2} + \frac{1-\theta_0}{x} - \eta_\infty - \frac{1}{x-y},$$
$$q(x,t) = \frac{\eta_\infty(\theta_0+\theta_\infty)}{2x} - \frac{t\mathcal{H}'_{D_6}}{x^2} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{yx}{t(x-y)},$$
$$b(x,t) = -\frac{y^2z}{t(x-y)},$$
$$t\mathcal{H}'_{D_6} = y^2z^2 - \{\eta_\infty y^2 + \theta_0 y - \eta_0 t\}z + \frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)y,$$
$$\alpha = -4\eta_\infty\theta_\infty, \beta = 4\eta_0(1+\theta_0), \gamma = 4\eta_\infty^2, \delta = -4\eta_0^2.$$

Painlevé III$'\big(D_7^{(1)}\big)$: the case $\gamma = 0$.

$$p(x,t) = \frac{\eta_0 t}{x^2} + \frac{1-\theta_0}{x} - \frac{1}{x-y},$$
$$q(x,t) = -\frac{t\mathcal{H}'_{D_7}}{x^2} + \frac{\theta_\infty}{2x} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{yx}{t(x-y)},$$
$$b(x,t) = -\frac{y^2z}{t(x-y)},$$
$$t\mathcal{H}'_{D_7} = y^2z^2 + (-\theta_0 y + \eta_0 t)z + \frac{\theta_\infty}{2}y,$$
$$\alpha = -4\theta_\infty, \beta = 4(\theta_0+1)\eta_0, \gamma = 0, \delta = -4\eta_0^2.$$

Painlevé III$'\big(D_7^{(1)}\big)$-2: the case $\delta = 0$.

$$p(x,t) = -\eta_\infty + \frac{1}{x} - \frac{1}{x-y},$$
$$q(x,t) = \frac{\theta_0 t}{2x^3} + \frac{\theta_\infty\eta_\infty}{2x} - \frac{t\mathcal{H}_{D_7\text{-}2}}{x^2} + \frac{yz}{x(x-y)},$$
$$a(x,t) = \frac{yx}{t(x-y)},$$
$$b(x,t) = -\frac{y^2z}{t(x-y)},$$
$$t\mathcal{H}_{D_7\text{-}2} = y^2z^2 - \eta_\infty y^2 z + \frac{\theta_\infty\eta_\infty}{2}y + \frac{\theta_0 t}{2y},$$
$$\alpha = -4\theta_\infty\eta_\infty, \beta = 4\theta_0, \gamma = 4\eta_\infty^2, \delta = 0.$$

Painlevé III$'\big(D_8^{(1)}\big)$:

$$
\begin{aligned}
&p(x,t) = \frac{2}{x} - \frac{1}{x-y},\\
&q(x,t) = \frac{t}{2x^3} - \frac{t\mathcal{H}'_{D_8}}{x^2} + \frac{1}{2x} + \frac{yz}{x(x-y)},\\
&a(x,t) = \frac{yx}{t(x-y)},\\
&b(x,t) = -\frac{y^2z}{t(x-y)},\\
&t\mathcal{H}'_{D_8} = y^2z^2 + yz + \frac{y}{2} + \frac{t}{2y},\\
&\alpha = -4, \beta = 4, \gamma = 0, \delta = 0.
\end{aligned}
$$

Painlevé V:

$$
\begin{aligned}
&p(x,t) = \frac{1-\kappa_0}{x} + \frac{\eta t}{(x-1)^2} + \frac{1-\theta}{x-1} - \frac{1}{x-y},\\
&q(x,t) = \frac{\kappa}{x(x-1)} - \frac{t\mathcal{H}_{\mathrm{V}}}{x(x-1)^2} + \frac{y(y-1)z}{x(x-1)(x-y)},\\
&a(x,t) = \frac{y-1}{t}\frac{x(x-1)}{x-y},\\
&b(x,t) = -\frac{y(y-1)^2z}{t(x-y)},\\
&t\mathcal{H}_{\mathrm{V}} = y(y-1)^2z^2 - \{\kappa_0(y-1)^2 + \theta y(y-1) - \eta t y\}z + \kappa(y-1),\\
&\alpha = \frac{1}{2}\kappa_\infty^2, \beta = -\frac{1}{2}\kappa_0^2, \gamma = (1+\theta)\eta, \delta = -\frac{1}{2}\eta^2,\\
&\kappa = \frac{1}{4}(\kappa_0+\theta)^2 - \frac{1}{4}\kappa_\infty^2.
\end{aligned}
$$

Painlevé deg-V:

$$
\begin{aligned}
&p(x,t) = \frac{1}{x-1} + \frac{1-\kappa_0}{x} - \frac{1}{x-y},\\
&q(x,t) = \frac{\gamma t}{2(x-1)^3} + \frac{\kappa}{x(x-1)} - \frac{t\mathcal{H}_{\mathrm{Vd}}}{x(x-1)^2} + \frac{y(y-1)z}{x(x-1)(x-y)},\\
&a(x,t) = \frac{y-1}{t}\frac{x(x-1)}{x-y},\\
&b(x,t) = -\frac{y(y-1)^2z}{t(x-y)},\\
&t\mathcal{H}_{\mathrm{Vd}} = y(y-1)^2z^2 - \kappa_0(y-1)^2z + \kappa(y-1) + \frac{\gamma t y}{2(y-1)},\\
&\alpha = \frac{1}{2}\kappa_\infty^2, \beta = -\frac{1}{2}\kappa_0^2, \delta = 0, \kappa = -\frac{1}{2}(\alpha+\beta) = \frac{1}{4}\big(\kappa_0^2 - \kappa_\infty^2\big).
\end{aligned}
$$

Painlevé VI:

$$
\begin{aligned}
p(x,t) &= \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \frac{1}{x-y},\\
q(x,t) &= \frac{\kappa}{x(x-1)} - \frac{t(t-1)\mathcal{H}_{\mathrm{VI}}}{x(x-1)(x-t)} + \frac{y(y-1)z}{x(x-1)(x-y)},\\
a(x,t) &= \frac{y-t}{t(t-1)}\frac{x(x-1)}{x-y}, \qquad b(x,t) = -\frac{y(y-1)(y-t)z}{t(t-1)(x-y)},\\
t(t-1)\mathcal{H}_{\mathrm{VI}} &= y(y-1)(y-t)z^2\\
&\quad - \{\kappa_0(y-1)(y-t) + \kappa_1 y(y-t) + (\theta-1)y(y-1)\}z\\
&\quad + \kappa(y-t),\\
\alpha &= \frac{1}{2}\kappa_\infty^2, \beta = -\frac{1}{2}\kappa_0^2, \gamma = \frac{1}{2}\kappa_1^2, \delta = \frac{1}{2}(1-\theta^2),\\
\kappa &= \frac{1}{4}(\kappa_0+\kappa_1+\theta-1)^2 - \frac{1}{4}\kappa_\infty^2.
\end{aligned}
$$

4.2. Degeneration

In this section, we list up all degenerations of the Painlevé equations and extended linear equations L_J. Here we consider degeneration of the extended linear system, which includes a deformation equation. In some cases, we should take a change of the dependent variable $u \to f(x,t,\varepsilon)u$. In [8], Okamoto did not treat the extended linear equations. If $b(x,t)$ is changed up to a function $r(t)$ in the limit, we denote $b \to b + r$.

P6 $\to$ P5: we change the variables

$$
\begin{aligned}
&t \to 1+\varepsilon t, \qquad \kappa_1 \to \varepsilon^{-1}\eta + \theta + 1, \qquad \theta \to -\varepsilon^{-1}\eta\\
&(\alpha \to \alpha, \beta \to \beta, \gamma \to -\delta\varepsilon^{-2} + \gamma\varepsilon^{-1}, \delta \to \delta\varepsilon^{-2}).
\end{aligned}
$$

In the limit $\varepsilon \to 0$, L_{VI} goes to L_{V} and

$$\mathcal{H}_{\mathrm{VI}} \to \varepsilon^{-1}\mathcal{H}_{\mathrm{V}} + O(\varepsilon^0) \qquad (\varepsilon \to 0).$$

P5 $\to$ deg-P5: we change the variables

$$z \to z + \frac{\gamma}{2\varepsilon(y-1)}, \eta \to \varepsilon, \theta \to \gamma\varepsilon^{-1}.$$

Then

$$\mathcal{H}_{\mathrm{V}} + \frac{\theta^2}{4t} \to \mathcal{H}_{\mathrm{Vd}} + O(\varepsilon^1) \qquad (\varepsilon \to 0).$$

For L_{V}, we first change

$$u \to (x-1)^{\theta/2}u$$

and $b \to b - \theta(y-1)/(2t)$. Then L_{V} goes to L_{Vd} in the limit $\varepsilon \to 0$.

P5 $\to$ P4: we change the variables

$$
\begin{aligned}
&t \to 1 + \sqrt{2}\varepsilon t, \ y \to \tfrac{\varepsilon}{\sqrt{2}}y, \ z \to \sqrt{2}\varepsilon^{-1}z, \ x \to \tfrac{\varepsilon}{\sqrt{2}}x,\\
&\kappa_\infty \to \varepsilon^{-2}, \theta \to \varepsilon^{-2} + 2\theta_\infty - \kappa_0, \eta \to -\varepsilon^{-2}\\
&(\alpha \to \varepsilon^{-4}/2, \beta \to \beta/4, \gamma \to -\varepsilon^{-4}, \delta \to -\varepsilon^{-4}/2 + \alpha\varepsilon^{-2}).
\end{aligned}
$$

After changing variables, we set $u \to \exp(\varepsilon^{-1}t/\sqrt{2})u$. Then in the limit $\varepsilon \to 0$, L_{V} goes to L_{IV} with $b \to b + t + y/2$ and

$$\sqrt{2}\left(\mathcal{H}_{\mathrm{V}} + \frac{(\kappa_0+\theta)^2 - \kappa_\infty^2}{4}\right) \to \varepsilon^{-1}\left(\mathcal{H}_{\mathrm{IV}} + 2\theta_\infty t\right) + O(\varepsilon^0) \qquad (\varepsilon \to 0).$$

P5 $\to$ P3$'\big(D_6^{(1)}\big)$: we change the variables

$$y \to 1+\varepsilon y,\, z \to z/\varepsilon,\, x \to 1+\varepsilon x,$$
$$\kappa_0 \to \varepsilon^{-1}\eta_\infty,\, \kappa_\infty \to \varepsilon^{-1}\eta_\infty - \theta_\infty,\, \theta \to \theta_0,\, \eta \to \varepsilon\eta_0$$
$$\left(\alpha \to \frac{1}{8}\varepsilon^{-2}\gamma + \frac{1}{4}\varepsilon^{-1}\alpha,\, \beta \to -\frac{\varepsilon^{-2}\gamma}{8},\, \gamma \to \frac{\varepsilon\beta}{4},\, \delta \to \frac{\varepsilon^2\delta}{8}\right).$$

In the limit $\varepsilon \to 0$, $L_{\rm V}$ goes to L_{D_6} and

$$\mathcal{H}_{\rm V} \to \mathcal{H}'_{D_6} + O(\varepsilon^1) \qquad (\varepsilon \to 0).$$

deg-P5 $\to$ P3$'\big(D_7^{(1)}\big)$-2: we change the variables

$$y \to 1+\varepsilon y,\, z \to \varepsilon^{-1}z,\, x \to 1+\varepsilon x,$$
$$\kappa_\infty \to \varepsilon^{-1}\eta_\infty,\, \kappa_0 \to \varepsilon^{-1}\eta_\infty + \theta_\infty,\, \gamma \to \theta_0\varepsilon/4$$
$$(\alpha \to -\varepsilon^{-2}\gamma/8,\, \beta \to \varepsilon^{-1}\alpha/4,\, \gamma \to \beta\varepsilon/4).$$

In the limit $\varepsilon \to 0$, $L_{\rm Vd}$ goes to $L_{D_7\text{-}2}$ and

$$\mathcal{H}_{\rm Vd} \to \mathcal{H}_{D_7\text{-}2} + O(\varepsilon) \qquad (\varepsilon \to 0).$$

deg-P5 $\to$ P34: we change the variables

$$t \to 1+\sigma\varepsilon^2 t,\, y \to \varepsilon^2 y,\, z \to z\varepsilon^{-2},\, x \to \varepsilon^2 x,$$
$$\kappa_\infty \to \sigma\sqrt{-2}\varepsilon^{-3},\, \gamma \to \varepsilon^{-6}$$
$$(\alpha \to -\varepsilon^{-6},\, \beta \to -\alpha/2,\, \gamma \to \varepsilon^{-6}).$$

In the limit $\varepsilon \to 0$, $L_{\rm Vd}$ goes to L_{34} and

$$\mathcal{H}_{\rm Vd} = \varepsilon^{-2}(\sigma\mathcal{H}_{34} - t^2/2) + \sigma t\varepsilon^{-4}/2 - \varepsilon^{-6}/2 + O(\varepsilon^{-1}) \qquad (\varepsilon \to 0).$$

P3$'$ $\to$ P3: this transformation is algebraic and we do not take any limit. If we change the variables

$$t \to t^2,\, y \to ty,\, z \to z/t,\, x \to tx,$$

L'_J is changed to L_J with $b \to b + \eta_\infty y$ if $J = D_6^{(1)}$ and

$$\mathcal{H}'_J = \frac{1}{2t}\mathcal{H}_J + \frac{yz}{2t^2},$$

for $J = D_6^{(1)}, D_7^{(1)}, D_7^{(1)}{-}2, D_8^{(1)}$.

P3$'\big(D_6^{(1)}\big) \to$ P3$'\big(D_7^{(1)}\big)$: we change the parameters

$$\eta_\infty \to \varepsilon,\, \theta_\infty \to \theta_\infty\varepsilon^{-1}.$$

In the limit $\varepsilon \to 0$, L'_{D_6} goes to L'_{D_7} and

$$\mathcal{H}'_{D_6} = \mathcal{H}'_{D_7} + O(\varepsilon) \qquad (\varepsilon \to 0).$$

P3$\big(D_6^{(1)}\big) \to$ P3$\big(D_7^{(1)}\big)$ is the same.

P3$'\big(D_6^{(1)}\big) \to$ P3$'\big(D_7^{(1)}\big)$-2: we change the parameters

$$\eta_0 \to \varepsilon,\, \theta_0 \to \theta_0\varepsilon^{-1},\, z \to z + \theta_0/(2\varepsilon y).$$

Then we have

$$\mathcal{H}'_{D_6} + \frac{\theta_0^2}{4t} \to \mathcal{H}'_{D_7\text{-}2} + O(\varepsilon) \qquad (\varepsilon \to 0).$$

For L'_{D_6}, we first change

$$u \to x^{\theta_0/2} u.$$

Then L'_{D_6} goes to $L'_{D_7\text{-}2}$. $\mathrm{P3}\big(D_6^{(1)}\big) \to \mathrm{P3}\big(D_7^{(1)}\big)$-2 is the same.

$\mathrm{P3}'\big(D_7^{(1)}\big) \to \mathrm{P3}'\big(D_7^{(1)}\big)$-2: this transformation is algebraic and we do not take any limit. If we change the variables

$$\begin{aligned} &y \to t/y, z \to (\theta_\infty y/2 - y^2 z)/t, \\ &\theta_0 \to \theta_\infty - 1, \theta_\infty \to \theta_0, \eta_0 \to \eta_\infty \\ &(\alpha \to -\beta, \beta \to -\alpha, \delta \to -\gamma), \end{aligned}$$

we have

$$\mathcal{H}'_{D_7} = \mathcal{H}'_{D_7\text{-}2} - \frac{yz}{t} - \frac{\theta_\infty(\theta_\infty - 2)}{4t}.$$

For L'_{D_7}, we first change

$$u \to u x^{(\theta_0+1)/2}.$$

Changing the variable $x \to t/x$, L_{D_7} is changed to $L'_{D_7\text{-}2}$ with $b \to b - yz/t$.

$\mathrm{P3}'\big(D_7^{(1)}\big) \to \mathrm{P3}'\big(D_8^{(1)}\big)$: we change the variables

$$\begin{aligned} &t \to 2t, y \to 2y, z \to z/2 + 1/(4\varepsilon y), \\ &\eta_0 \to \varepsilon, \theta_0 \to -1 + \varepsilon^{-1}, \theta_\infty \to 1/2. \end{aligned}$$

Then

$$\mathcal{H}'_{D_7} = \frac{1}{2}\mathcal{H}'_{D_8} - \frac{1}{8\varepsilon^2 t} + \frac{1}{4\varepsilon t} + O(\varepsilon) \qquad (\varepsilon \to 0).$$

For L'_{D_7}, we first change

$$u \to x^{(1+\theta_0)/2} u.$$

Changing the variable $x \to 2x$, L'_{D_7} goes to L'_{D_8} in the limit $\varepsilon \to 0$. $\mathrm{P3}\big(D_7^{(1)}\big) \to \mathrm{P3}\big(D_8^{(1)}\big)$ is the same.

$\mathrm{P3}'\big(D_7^{(1)}\big)$-2 $\to \mathrm{P3}'\big(D_8^{(1)}\big)$: we change the variables

$$\begin{aligned} &t \to -2t, z \to z + 1/(2y), \\ &\eta_\infty \to \varepsilon, \theta_0 \to -1/2, \theta_\infty \to \varepsilon^{-1}. \end{aligned}$$

Then

$$\mathcal{H}'_{D_7\text{-}2} = -\frac{1}{2}\mathcal{H}'_{D_8} - \frac{1}{8t} + O(\varepsilon) \qquad (\varepsilon \to 0).$$

For $L'_{D_7\text{-}2}$, we first change

$$u \to x^{1/2} u.$$

Changing the variables, $L'_{D_7\text{-}2}$ goes to L'_{D_8} in the limit $\varepsilon \to 0$. $\mathrm{P3}\big(D_7^{(1)}\big)$-2 $\to \mathrm{P3}\big(D_8^{(1)}\big)$ is the same.

$\mathrm{P3}\big(D_6^{(1)}\big) \to \mathrm{P2}$: we change the variables

$$\begin{aligned} &t \to 1 + \varepsilon^2 t, y \to 1 + 2\varepsilon y, z \to 1 + \varepsilon^{-1} z/2, x \to 1 + 2\varepsilon x, \\ &\eta_0 \to -\varepsilon^{-3}/4, \eta_\infty \to \varepsilon^{-3}/4, \theta_0 \to -\varepsilon^{-3}/2 - 2\alpha - 1, \theta_\infty \to \varepsilon^{-3}/2 \\ &\left(\alpha \to -\frac{\varepsilon^{-6}}{2}, \beta \to \frac{1}{2}\varepsilon^{-6}(1 + 4\alpha\varepsilon^3), \gamma \to \frac{\varepsilon^{-6}}{4}, \delta \to -\frac{\varepsilon^{-6}}{4}\right). \end{aligned}$$

After changing variables, we set $u \to \exp(-\varepsilon^{-1}t/4)u$. Then in the limit $\varepsilon \to 0$, L_{D_6} goes to L_{II} and

$$\mathcal{H}_{D_6} + \eta_0(\theta_0 + \theta_\infty) \to \varepsilon^{-2}\mathcal{H}_{\mathrm{II}} + O(\varepsilon^{-1}) \qquad (\varepsilon \to 0).$$

P4 → P4_34: this transformation is algebraic and we do not take any limit. We change the variables

$$t \to \sigma\varepsilon t - \varepsilon^{-1}\theta/2,\ y \to 2\varepsilon y,\ z \to \varepsilon^{-1}z/2,$$
$$x \to 2\varepsilon x,\ \theta_\infty \to (\kappa_0 - 1)/2 + \varepsilon^{-2}\theta^2/8$$
$$(\alpha \to \varepsilon^{-6}\beta^2/16,\ \beta \to -2\alpha,\ 2\varepsilon^4 \to \gamma).$$

Then L_{IV} is changed to L_{4_34} with $b \to b - \sigma\varepsilon^2 y - \varepsilon^2 t + \sigma\theta/2$ and

$$\mathcal{H}_{\mathrm{IV}} = \sigma\varepsilon^{-1}\mathcal{H}_{4\_34}$$

for $\eta = 2\varepsilon^2$.

P4 → P2: we change the variables

$$t \to -\varepsilon^{-3}(1 - 2^{-2/3}\varepsilon^4 t),\ y \to \varepsilon^{-3}(1 + 2^{2/3}\varepsilon^2 y),\ z \to 2^{-2/3}\varepsilon z,$$
$$x \to \varepsilon^{-3}(1 + 2^{2/3}\varepsilon^2 x),\ u \to \exp(2^{-5/3}\varepsilon^{-2}t)u,$$
$$\kappa_0 \to \varepsilon^{-6}/2,\ \theta_\infty \to -\alpha - 1/2 \quad \left(\alpha \to -2\alpha - \frac{1}{2\varepsilon^6},\ \beta \to -\frac{1}{2\varepsilon^{12}}\right).$$

In the limit $\varepsilon \to 0$, L_{IV} goes to L_{II} and

$$\mathcal{H}_{\mathrm{IV}} \to 2^{2/3}\varepsilon^{-1}\mathcal{H}_{\mathrm{II}} - \varepsilon^{-3}(\alpha + 1/2) + O(\varepsilon^0) \qquad (\varepsilon \to 0).$$

P4 → P34: we change the variables

$$t \to \varepsilon t + \sigma\varepsilon^{-3}/4,\ y \to 2\sigma\varepsilon y,\ z \to \sigma\varepsilon^{-1}z/2 + \sigma\varepsilon^{-3}/8,$$
$$x \to \sigma\varepsilon x,\ u \to \exp\frac{\sigma x}{8\varepsilon^3}u,$$
$$\theta_\infty \to \varepsilon^{-6}/32 \qquad (\alpha \to \varepsilon^{-6}/16,\ \beta \to -2\alpha).$$

In the limit $\varepsilon \to 0$, L_{IV} goes to L_{34} and

$$\mathcal{H}_{\mathrm{IV}} \to \frac{1}{\varepsilon}\mathcal{H}_{34} - \frac{\sigma\kappa_0}{4\varepsilon^3} + O(\varepsilon^0) \qquad (\varepsilon \to 0).$$

P3$\left(D_7^{(1)}\right)$ → P1: we change the variables

$$t \to (\varepsilon^{-10} + \varepsilon^{-6}t)/2,\ y \to 1 + 2\varepsilon^2 y,\ z \to \varepsilon^{-2}z/2 - \frac{\varepsilon^{-5}}{4\sqrt{-2}} - \frac{\varepsilon^{-3}y}{2\sqrt{-2}},$$
$$\theta_0 \to -1 + 3\sqrt{-2}\varepsilon^{-5}/4,\ \theta_\infty \to -1/2,\ \eta_0 \to \sqrt{-2}\varepsilon^5$$
$$(\alpha \to 2,\ \beta \to -6,\ \delta \to 8\varepsilon^{10}).$$

Then

$$\mathcal{H}_{D_7} - \frac{5}{8}\eta_0^2 t - \frac{\eta_0}{4} - \frac{1}{4} \to 2\varepsilon^6\mathcal{H}_I + O(\varepsilon^7) \qquad (\varepsilon \to 0).$$

For L_{D_7}, we first change

$$u \to x^{(\theta_0-1)2}\exp\left(\frac{\eta_0 t}{2x} - \frac{3\eta_0 t}{4}\right)u.$$

Changing the variable

$$x \to 1 + 2\varepsilon^2 x,$$

L_{D_7} goes to L_I in the limit $\varepsilon \to 0$.

P3$\big(D_7^{(1)}\big)$-2 $\to$ P1: we change the variables

$$t \to (\varepsilon^{-10} + \varepsilon^{-6}t)/2,\ y \to 1 - 2\varepsilon^2 y,\ z \to -\varepsilon^{-2}z/2 - \varepsilon^{-5}/(2\sqrt{-2}),$$
$$\theta_0 \to -1/2,\ \theta_\infty \to 1/2 - 3\varepsilon^{-5}/(2\sqrt{-2}),\ \eta_\infty \to \sqrt{-2}\varepsilon^5$$
$$(\alpha \to -6,\ \beta \to -2,\ \gamma \to -8\varepsilon^{10}).$$

Then

$$\mathcal{H}_{D_7\text{-}2} + \frac{\eta_\infty^2 t}{2} \to 2\varepsilon^6\,\mathcal{H}_I - 2 + O(\varepsilon^7) \qquad (\varepsilon \to 0).$$

For $L_{D_7\text{-}2}$, we first change

$$u \to x^{(\theta_0-1)2}\exp\left(\frac{2\theta_\infty x}{3} - \frac{3\eta_0 t}{2}\right)u.$$

Changing the variable

$$x \to 1 - 2\varepsilon^2 x,$$

$L_{D_7\text{-}2}$ goes to L_I in the limit $\varepsilon \to 0$.
P34 $\to$ P1: we change the variables

$$t \to -\sigma\varepsilon^2 t + 6\sigma\varepsilon^{-10},\ y \to 2\varepsilon^{-4}y - 2\varepsilon^{-10},\ z \to \varepsilon^4 z/2 + \varepsilon y + \varepsilon^{-5},$$
$$\kappa_0 \to -4\varepsilon^{-15} \qquad (\alpha \to 16\varepsilon^{-30}).$$

Then

$$\mathcal{H}_{34} \to -\sigma\varepsilon^{-2}\mathcal{H}_I + 6\sigma\varepsilon^{-20} - \sigma\varepsilon^{-8}t + O(\varepsilon^{-1}) \qquad (\varepsilon \to 0).$$

For L_{34}, we first change

$$u \to x^{\kappa_0/2}u.$$

Changing the variable

$$x \to 2\varepsilon^{-4}x - 2\varepsilon^{-10},$$

L_{34} goes to L_I in the limit $\varepsilon \to 0$.
P1_2 $\to$ P2: this transformation is algebraic and we do not take any limit. We change the variables

$$t \to \varepsilon^{-2}t + 6\varepsilon^{-2}\theta^2,\ y \to \varepsilon^{-1}y - \varepsilon^{-1}\theta,\ z \to \varepsilon z - 2\varepsilon\theta y + 4\varepsilon\theta^2,$$
$$\eta = \varepsilon^3,\ \beta = \varepsilon^5\theta \quad (\alpha = \varepsilon^6,\ \beta = \varepsilon^5\theta).$$

Then

$$\mathcal{H}_{1\_2} = \varepsilon^2\left(\mathcal{H}_{\mathrm{II}} + \frac{\theta}{2} - \theta^2 t\right)$$

for $\alpha = 4\theta^3$ in P2. For L_{1_2}, we first change

$$u \to \exp\left(-\frac{\beta x^2}{\eta} + \frac{2\beta^2 x}{\eta^3}\right)u.$$

Changing the variable

$$x \to \varepsilon^{-1}x - \varepsilon^{-1}\theta,$$

L_{1_2} is changed to L_{II} for $\alpha = 4\theta^3$.

P2 → P1: we change the variables

$$t \to \varepsilon^2 t - 6\varepsilon^{-10},\ y \to \varepsilon y + \varepsilon^{-5},$$
$$z \to \varepsilon^{-1} z + (\varepsilon y + \varepsilon^{-5})^2 + (\varepsilon^2 t - 6\varepsilon^{-10})/2,\ \alpha \to 4\varepsilon^{-15}.$$

Then

$$\mathcal{H}_{\mathrm{II}} = \varepsilon^{-2}\mathcal{H}_{\mathrm{I}} - 6\varepsilon^{-20} + \varepsilon^{-8} t - \varepsilon^{-5}/2 + O(\varepsilon) \qquad (\varepsilon \to 0).$$

For L_{II}, we first change

$$u \to \exp\left(\frac{x^3}{3} + \frac{tx}{2}\right) u.$$

Changing the variable

$$x \to \varepsilon x + \varepsilon^{-5},$$

L_{II} goes to L_{I} in the limit $\varepsilon \to 0$.

Remark. In [8], there is a misprint in P3 → P2.

5. Isomonodromic deformations of SL-type

We will list up *five* types of isomonodromic deformations of SL-type. This section is the revision of section 4.4 in [8]. The isomonodromic deformation of SL-type is

$$\frac{\partial^2 u}{\partial x^2} = p(x,t)u, \qquad \frac{\partial u}{\partial t} = A(x,t)\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial A(x,t)}{\partial x}u.$$

The compatibility condition is given by

$$p_t(x,t) = 2p(x,t)A_x(x,t) + A(x,t)p_x(x,t) - \tfrac{1}{2}A_{xxx}(x,t).$$

In the following, $p(x,t)$ contains a Hamiltonian K. The compatibility condition is the Hamiltonian system with the Hamiltonian K. Eliminating z, we obtain the Painlevé equation on y, which is an apparent singularity of the linear equation. We remark that Fuchs studied the isomonodromic deformations of SL-type for P6 [2].

Type (4), (7/2): P1_2(α, β).

$$p(x,t) = \alpha x^4 + 4\beta x^3 + \alpha t x^2 + 2\beta t x + 2K_{1,2} + \frac{3}{4(x-y)^2} - \frac{z}{x-y},$$
$$A(x,t) = \frac{1}{2(x-y)},$$
$$K_{1,2} = \frac{z^2}{2} - \beta(2y^3 + ty) - \frac{\alpha}{2}(y^4 + ty^2).$$

Type (1)(3), (1)(5/2): P4_34$'(\alpha, \beta, \gamma)$ for $\sigma = 1$, P4_34(α, β, γ) for $\sigma = -1$.

$$p(x,t) = \frac{\gamma}{2}(x^2 + 2\sigma t x + t^2) + \frac{\beta}{2}(x + \sigma t) + \frac{\alpha - 1}{4x^2} + \frac{\sigma K_{4,34}}{x} + \frac{3}{4(x-y)^2} - \frac{yz}{x(x-y)},$$
$$A(x,t) = \frac{\sigma x}{x-y},$$
$$K_{4,34} = \sigma y z^2 - \sigma z + \frac{\sigma(1-\alpha)}{4y} - \frac{\beta}{2}y(\sigma y + t) - \frac{\sigma\gamma}{2}y(y + \sigma t)^2.$$

Type $(2)^2$, $(2)(3/2)$, $(3/2)^2$: $\mathrm{P3}'(\alpha,\beta,\gamma,\delta)$.

$$\begin{aligned}
p(x,t) &= \frac{a_0t^2}{x^4} + \frac{a_0't}{x^3} + \frac{tK_{\mathrm{III}}'}{x^2} + \frac{a_\infty'}{x} + a_\infty + \frac{3}{4(x-y)^2} - \frac{yz}{x(x-y)},\\
A(x) &= \frac{yx}{t(x-y)},\\
tK_{\mathrm{III}}' &= y^2z^2 - yz - \frac{a_0t^2}{y^2} - \frac{a_0't}{y} - a_\infty' y - a_\infty y^2,\\
a_0 &= -\frac{\delta}{16}, a_0' = -\frac{\beta}{8}, a_\infty = \frac{\gamma}{16}, a_\infty' = \frac{\alpha}{8}.
\end{aligned}$$

Type $(1)^2(2)$, $(1)^2(3/2)$: $\mathrm{P5}(\alpha,\beta,\gamma,\delta)$.

$$\begin{aligned}
p(x,t) = &\frac{a_1t^2}{(x-1)^4} + \frac{K_{\mathrm{V}}t}{(x-1)^2x} + \frac{a_2t}{(x-1)^3} - \frac{z(y-1)y}{x(x-1)(x-y)}\\
&+ \frac{a_\infty}{(x-1)^2} + \frac{a_0}{x^2} + \frac{3}{4(x-y)^2},
\end{aligned}$$

$$A(x) = \frac{y-1}{t}\cdot\frac{x(x-1)}{x-y},$$

$$tK_{\mathrm{V}} = y(y-1)^2\left[-\frac{a_1t^2}{(y-1)^4} - \frac{a_2t}{(y-1)^3} + z^2 - \left(\frac{1}{y} + \frac{1}{y-1}\right)z - \frac{a_\infty}{(y-1)^2} - \frac{a_0}{y^2}\right],$$

$$a_0 = -\frac{\beta}{2} - \frac{1}{4}, a_1 = -\frac{\delta}{2}, a_2 = -\frac{\gamma}{2}, a_\infty = \frac{1}{2}(\alpha+\beta) - \frac{3}{4}.$$

Type $(1)^4$: $\mathrm{P6}(\alpha,\beta,\gamma,\delta)$.

$$\begin{aligned}
p(x,t) = &\frac{a_0}{x^2} + \frac{a_1}{(x-1)^2} + \frac{a_\infty}{x(x-1)} + \frac{b_1}{(x-t)^2} + \frac{3}{4(x-y)^2}\\
&+ \frac{t(t-1)K_{\mathrm{VI}}}{x(x-1)(x-t)} - \frac{y(y-1)z}{x(x-1)(x-y)},
\end{aligned}$$

$$A(x) = \frac{y-t}{t(t-1)}\cdot\frac{x(x-1)}{x-y},$$

$$K_{\mathrm{VI}} = \frac{y(y-1)(y-t)}{t(t-1)}\left[z^2 - \left(\frac{1}{y} + \frac{1}{y-1}\right)z - \frac{a_0}{y^2} - \frac{a_1}{(y-1)^2} - \frac{a_\infty}{y(y-1)} - \frac{b_1}{(y-t)^2}\right],$$

$$a_0 = -\frac{\beta}{2} - \frac{1}{4}, a_1 = \frac{\gamma}{2} - \frac{1}{4}, b_1 = -\frac{1}{2}\delta, a_\infty = \frac{1}{2}(\alpha+\beta-\gamma+\delta-1).$$

If we set $\alpha = 0$ in P1_2, we obtain the standard isomonodromic deformations of SL-type for P1. If we set $\gamma = 0$ in P4_34, we obtain the SL-type for P34. If we set $\gamma = 0$ in $\mathrm{P3}'$, we obtain the SL-type for $\mathrm{P3}'\big(D_7^{(1)}\big)$. If we set $\gamma = 0, \delta = 0$ in $\mathrm{P3}'$, we obtain the SL-type for $\mathrm{P3}'\big(D_8^{(1)}\big)$. If we set $\delta = 0$ in P5, we obtain the SL-type for deg-P5.

We show the standard isomonodromic deformations of SL-type for P2 and P4.

$\mathrm{P2}(\alpha)$:

$$\begin{aligned}
p(x,t) &= x^4 + tx^2 + 2\alpha x + 2K_{\mathrm{II}} + \frac{3}{4(x-y)^2} - \frac{z}{x-y},\\
A(x,t) &= \frac{1}{2}\cdot\frac{1}{x-y},\\
K_{\mathrm{II}} &= \frac{1}{2}z^2 - \frac{1}{2}y^4 - \frac{1}{2}ty^2 - \alpha y.
\end{aligned}$$

P4(α, β):

$$p(x,t) = \frac{a_0}{x^2} + \frac{K_{\mathrm{IV}}}{2x} + a_1 + \left(\frac{x+2t}{4}\right)^2 + \frac{3}{4(x-y)^2} - \frac{yz}{x(x-y)},$$

$$A(x) = \frac{2x}{x-y},$$

$$K_{\mathrm{IV}} = 2yz^2 - 2z - \frac{2a_0}{y} - 2a_1 y - 2y\left(\frac{y+2t}{4}\right)^2,$$

$$a_0 = -\frac{\beta}{8} - \frac{1}{4}, a_1 = -\frac{\alpha}{4}.$$

Remark. In [8], there is a misprint in K_{IV}.

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